Extensions of Mapper–type algorithms and their applications to knot theory

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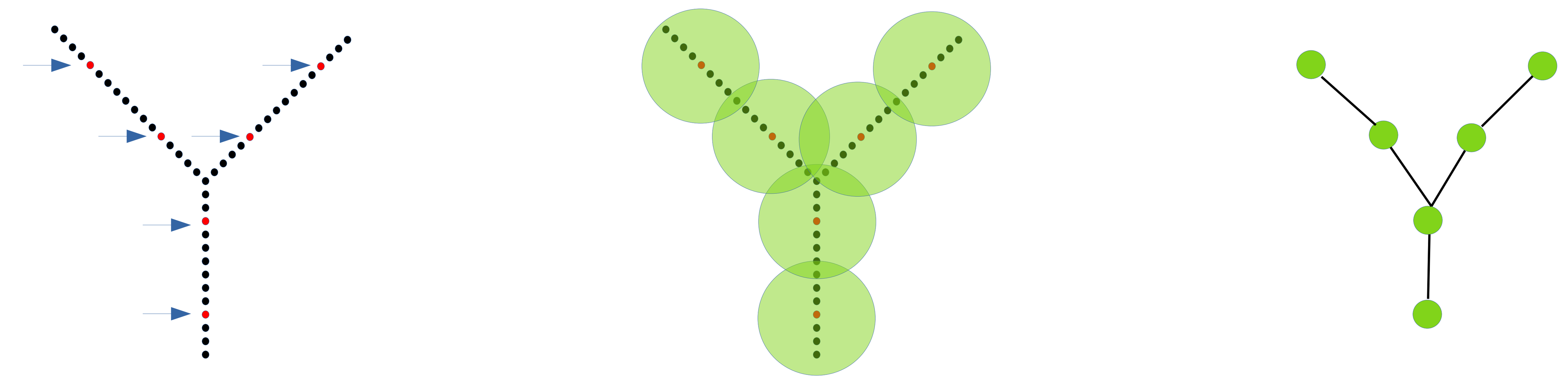
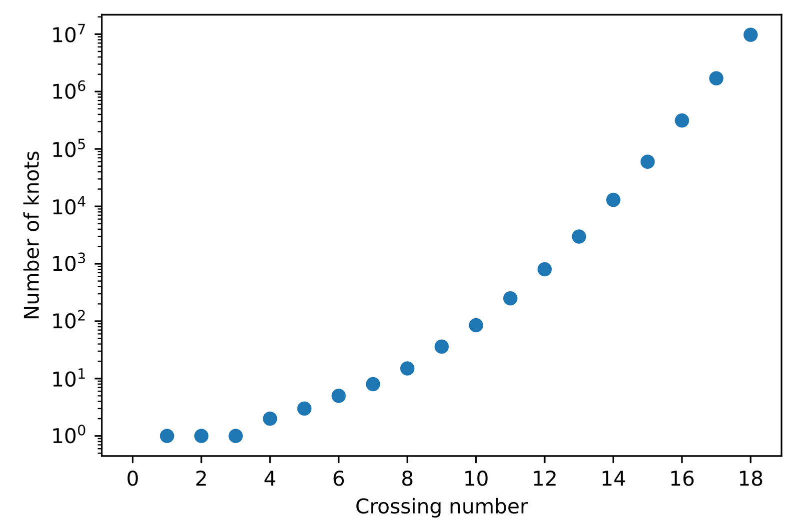
1. Ball Mapper



Mapper algorithms are tools aiming to visualize high dimensional datasets. Starting from a point cloud X, and a constant $\epsilon > 0$, we select a subset $L \subset X$ having the property that for every $x \in X$ there exist $l \in L$ such that $d(x,l) \leq \epsilon$ (left). L is called an ϵ**−net** of X. Note that $\bigcup_{l \in L} B(l,\epsilon)$ is an overlapping cover of X (middle). The **Ball Mapper graph** is obtained by assigning each ball $B(l,\epsilon)$ with a vertex $v(l)$ of the graph, and by placing an edge between any two vertices whose corresponding balls jointly cover points from X (right).

2. Knots

A knot is an embedding of S^1 into $\mathbb{R}^3$ up to isotopy. Knot invariants have the same value on isotopic knots, but sometimes fails to distinguish between non-isotopic knots. In this work we focus on **polynomial invariants** such as the Alexander, Jones, HOMFLY-PT and Khovanov Q polynomials.



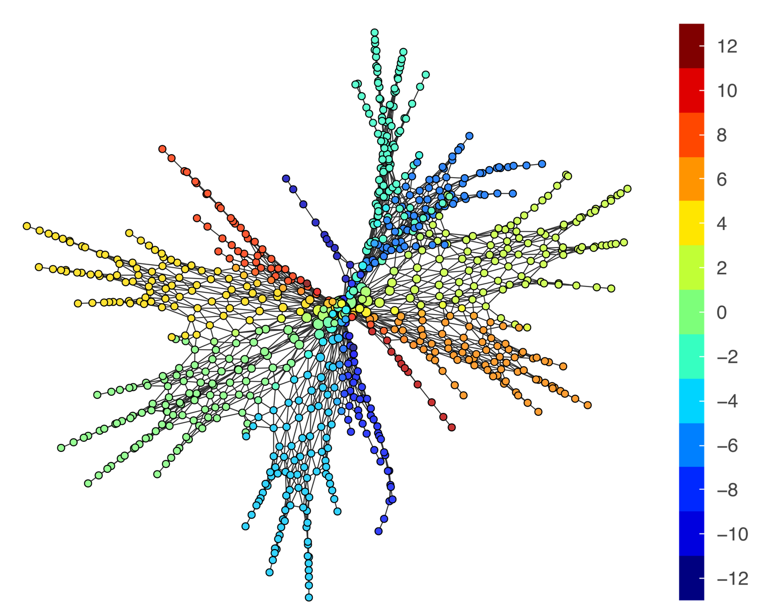
3. From knots to pointclouds

Given a finite collection of knots $\mathcal{K}$, we construct a point cloud $\mathcal{I}(\mathcal{K})$ corresponding to the coefficients of a polynomial invariant $\mathcal{I}$. For each knot $K \in \mathcal{K}$ we extract a **vector of the coefficients** and pad it with zeros to ensure correct alignment. The length of the vectors is $max_t - min_t + 1$, where min_t, max_t are the minimal and maximal powers. For two-variable polynomials we flatten out the coefficient matrix.

4. Vectorization example

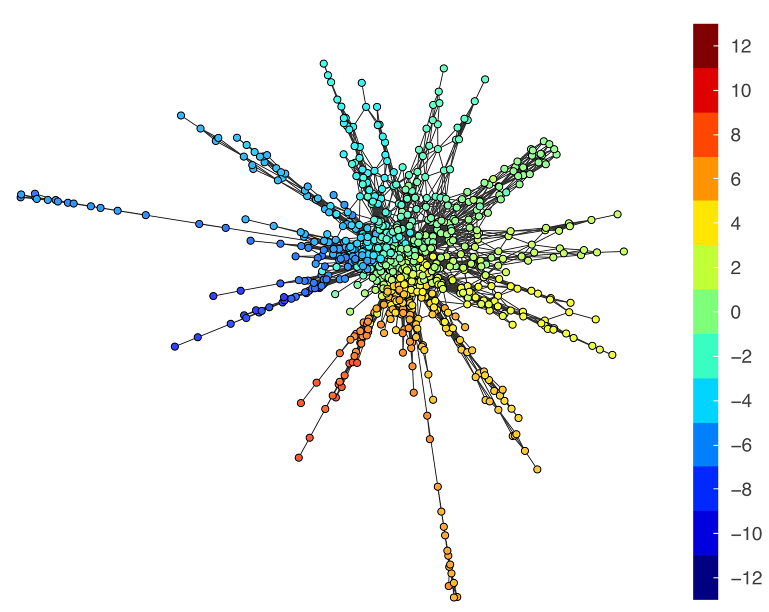
	Trefoil	Data vector	Dim
Alexander	$t^{-1} - 1 + t$	(0,1,-1,1,0)	17
Jones	$t + t^3 - t^4$	(0,0,0,0,0,1,0,1,-1)	51
HOMFLY	$-a^4 + 2a^2 + a^2z^2$	(-1,0,0,0,0,0,2,0,0,**0,0,0,0,0,0,1,0,0**)	152
Khovanov	$q^{-9}t^{-3} + q^{-5}t^{-2} + q^{-3} + q^{-1}$	(0,0,0,1,0,0,0,**0,0,0,1,0,0,0,**0,1,0,0,0,0,0, **0,0,0,0,0,0,0,**1,0,0,0,0,0,0)	3003

5. Let's color by signature

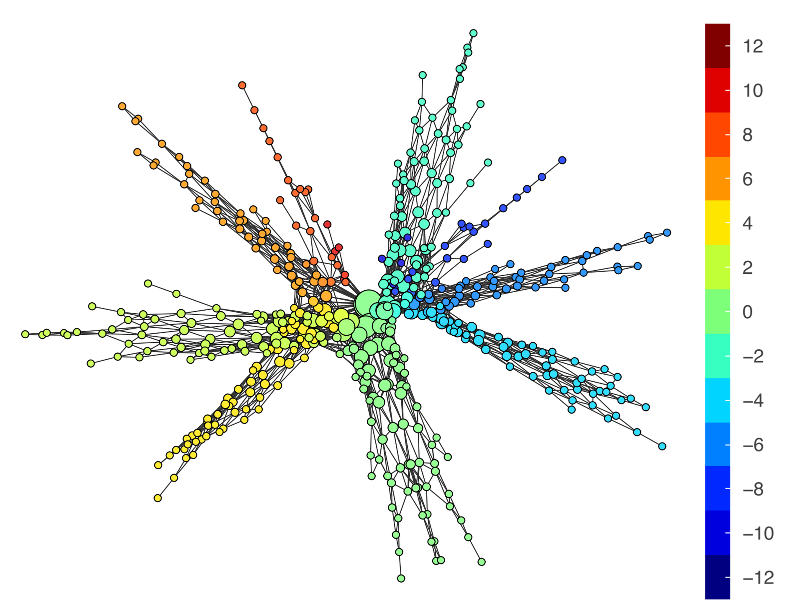


Jones

Alexander (mod 4)



HOMFLY-PT



Khovanov

6. Equivariant Ball Mapper

Given a point cloud X and an automorphism group of isometries H acting on it, we modify the Ball Mapper algorithm in such a way that there is a **lift of an action** of H to the Ball Mapper graph G. For every point $x \in X$, and isometry $g \in H$ the *orbit* $\Omega_g(x)$ of x will contains the sequence of points $x, g(x), g^2(x), \ldots, g^n(x) \in X$. We assume that $g^{n+1}(x) = x$.

To obtain an equivariant Ball Mapper is sufficient to require the ϵ-net L to be invariant under the action of H. This can be obtained by adding the whole orbit $\Omega_g(l) = \{g(l)\}_{g \in H}$ to the set of landmark points each time a new landmark l is added.

7. Mapper on Ball Mapper

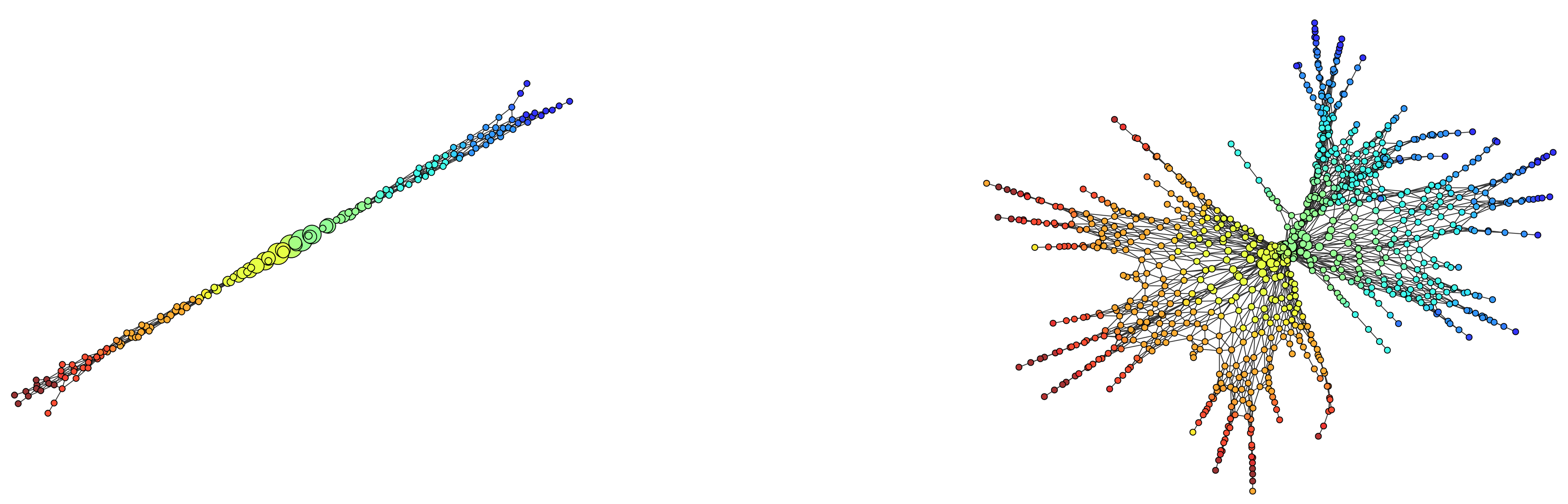
The (conventional) Mapper construction is based on a interval cover $\mathcal{C}$ of the range of the **lens function** $f : X \to \mathbb{R}^n$. Typically $n = 1$ or is a small number, as the range of f needs to be covered with a collection of overlapping intervals. However, lens functions having ranges in $\mathbb{R}^n$, for large value of n, are more likely to preserve important information about the high dimensional dataset X.

By considering $f(X)$ alone as a point cloud we can apply the Ball Mapper construction to it. Note that the Ball Mapper provides an overlapping, adaptive cover of $f(X)$ that can be used in the Mapper algorithm. This opens up the possibility to use higher dimensional lens functions.

8. Mapper-based visualization of functions

We can use BM to visualize functions $f : X \subset \mathbb{R}^n \to Y \subset \mathbb{R}^m$ for $m, n \gg 1$.

We build Ball Mapper graphs $G(X)$ and $G(Y)$ and look where points covered by a subset of vertices of $G(X)$ get mapped to $G(Y)$. This can be achieved via a custom coloring.



Representation of map from the space of Alexander (left) to Jones polynomials (right). Regions of the same color are mapped to each other.

References

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