Section 5
Classical & Discrete Morse Theory.

"Every mathematician has a secret weapon. Mine is Morse theory" Raoul Bott.

Let us start from a very beginning, from a basic calculus:

Let $f: M \to \mathbb{R}$ be continuous/smooth and M be compact.

Then $\exists \bar{x} \in M \mid f(\bar{x}) = \max_{x \in M} f(x)$ and

$$\exists \underline{x} \in M \mid f(\underline{x}) = \min_{x \in M} f(x).$$

Let us think about this theorem. Clearly $\bar{x}$ & $\underline{x}$ are critical points of f. Therefore we have a relation:

M - compact $\implies$ Any smooth function on f has at least two critical points

topological property — analytical property.

Morse Theory is very for generalization of this statement. 2

It establish a relation between critical cells of sufficiently smooth functions f on manifold M with topological properties of M. In particular, it gives a lower bound on the number and type of critical points of f as a function of homotopy type of M.

But, let us start from the beginning.

Let M be compact smooth closed manifold in $\mathbb{R}^n$

$f: M \to \mathbb{R}$, smooth.

$x \in M$ is critical if $\frac{\partial f}{\partial x_i} = 0$ for some i.

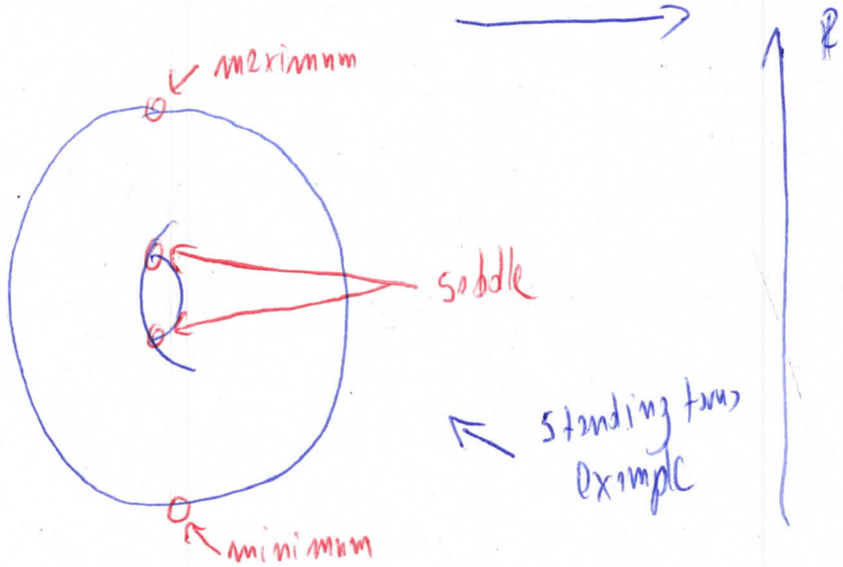
$x \in M$ is nondegenerated critical point if its Hessian matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{i,j=1 \dots n} \text{ is invertible}$$

A function f is a Morse function if all its critical points are not degenerated.

Let us look at a few examples;

Let M be a torus; f is this projection



← at every point, except from critical points, there is unique direction of decent

In this case in every point of a torus except from the four points marked with red at every point there is a unique direction of steepest decent. This is a positive example, and this is by far the most studied Morse function, as you can find it in any Morse theory book.

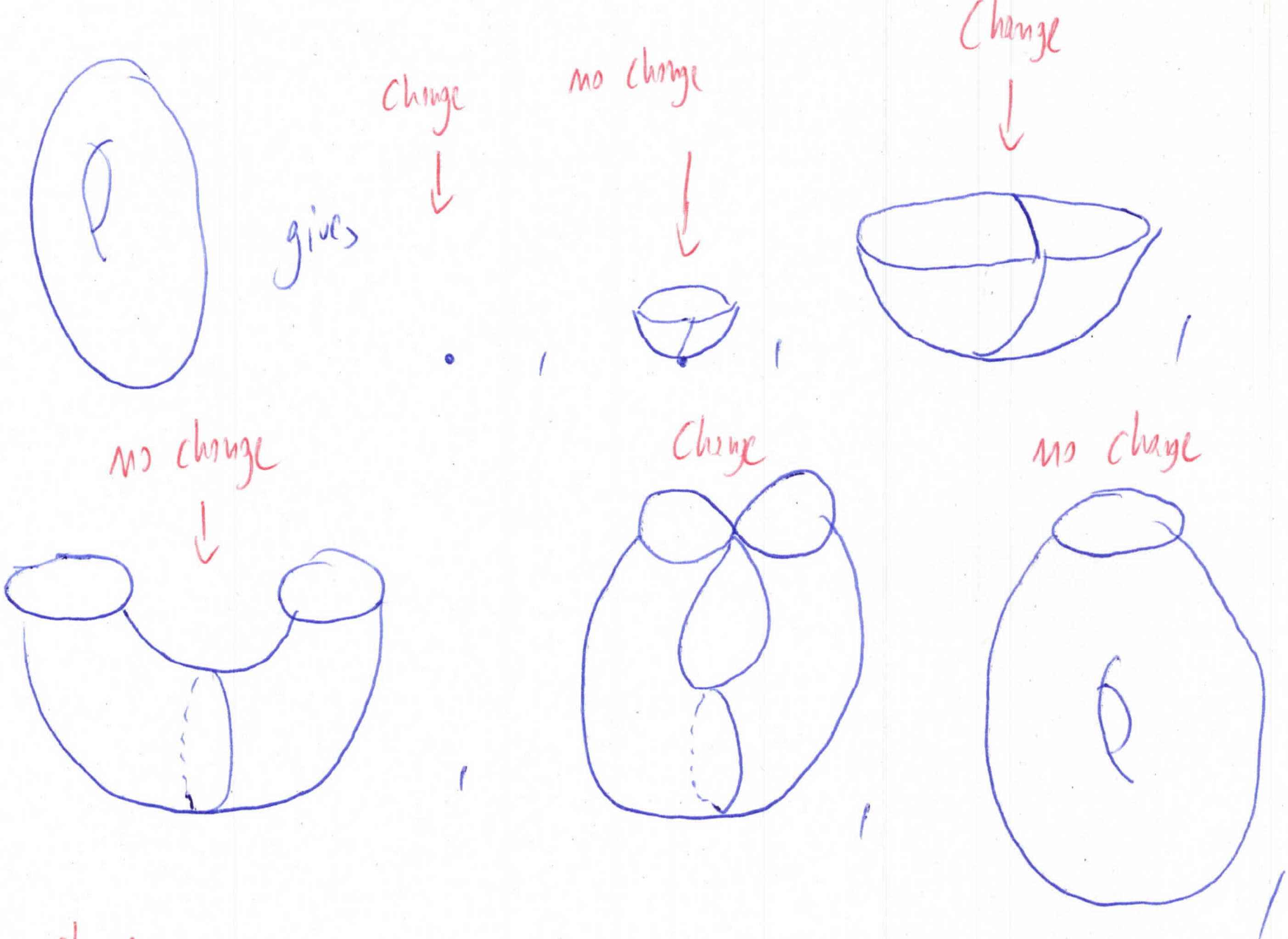
Let us now take a look at a different example:

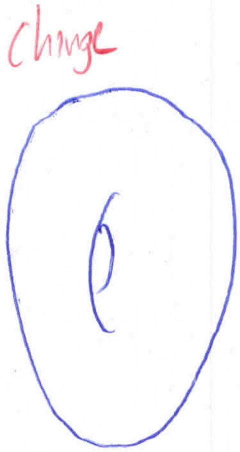
In this case the minima of function is a curve ○ & so is maxima.

All critical points are degenerated

lying torus example This is <u>not</u> a Morse function

Now let us suppose that we have a closed manifold M and a Morse function on it. Let us consider the change in topology of $f^{-1}((-\infty, a])$ when we vary a.





It is easy to see, at least in this example, that we have a change in topology of $f^{-1}((-\infty, a])$ only when a is a critical value i.e. $f(a)$ contains a critical point.

One of the Main result of a Discrete Morse Theory states this fact:

Theorem

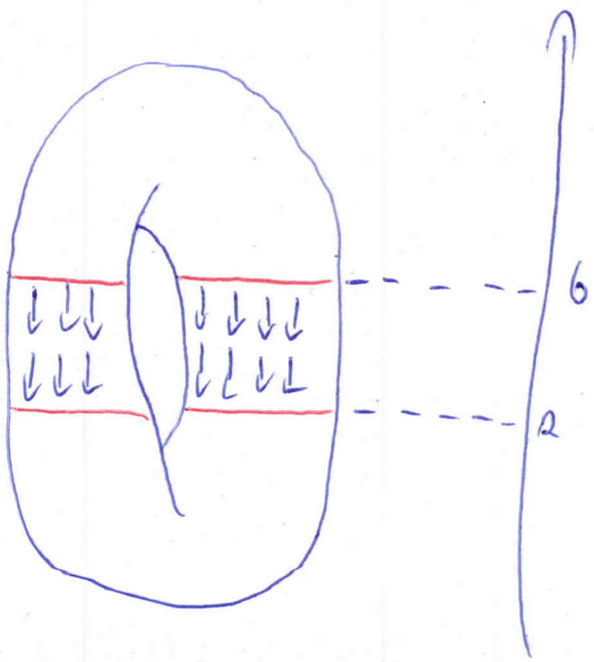
M - closed manifold, $f: M \to \mathbb{R}$ - Morse function.
Let us assume that $f^{-1}([a,b])$ have no critical points.

Then:

1. $f^{-1}((-\infty, a])$ is diffeomorphic to $f^{-1}([-\infty, b])$
2. $f^{-1}(a)$ — " — $f^{-1}(b)$
3. $f^{-1}([a,b])$ is diffeomorphic to $f^{-1}(a) \times [a,b]$.

The deformation is given by $\operatorname{grad}(f)$. As there are no critical points of f in $[a,b]$ this is uniquely defined.

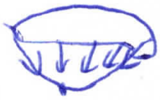
Eg



In fact, more is true. Moving through a critical point corresponding to gluing a handle of a dimension i where i is the <u>index of critical point</u> (i.e. number of descending directions).

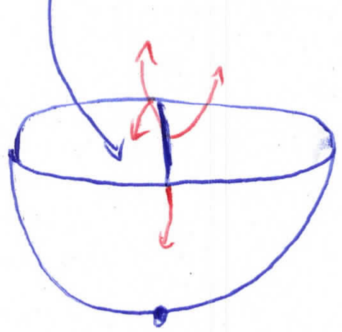
Let us look at our most classical example:

← no descending directions, $i=0$, 0d handle = point

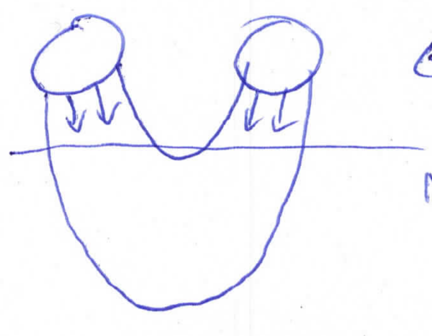


← no change in topology, this whole structure can be contracted to a point

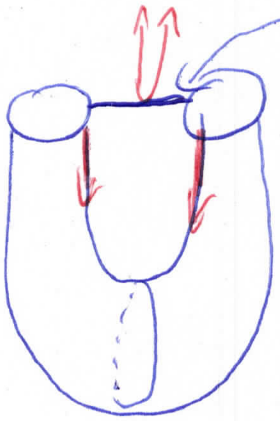
one dimensional handle



← One descending direction, adding one dimensional handle



← no change in topology, everything can be collapsed to this level using the gradient.



← One direction up, one direction down, adding one dimensional handle.

← no change in topology, everything can be contracted to this level.

← two descending directions, adding two dimensional handle (a cap)

Therefore we have a function, manifold & correspondence of critical points & homology change.

In ~~top~~ particular, homology change when moving by critical points.

Let us now build discrete analogy of this theory, where M is a regular CW-complex (in particular, simplicial or cubical complex) & $f : M \to \mathbb{R}$ is a function defined on cells of M.

A function f is called a Discrete Morse Function if for every cell $\sigma \in M$:

(*)

$$\text{cardinality}\{\tau > \sigma ;\ \dim(\tau) = \dim(\sigma)+1,\ f(\tau) \leq f(\sigma)\} \leq 1$$

number of cells τ in the coboundary of σ that are on the same level or lower then σ is at most 1

AND

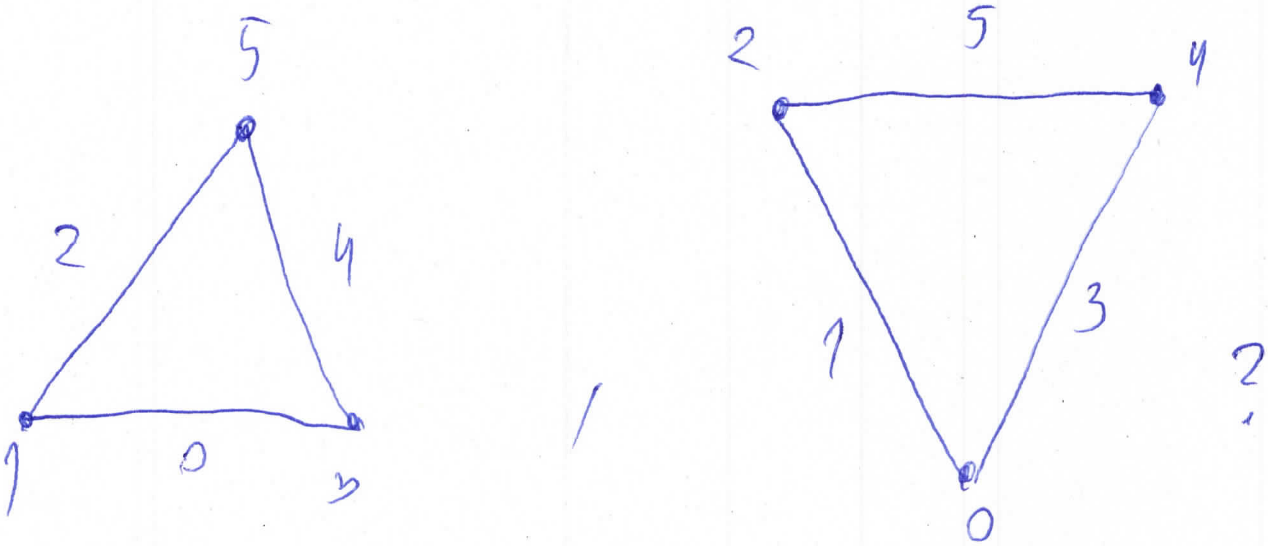
(**)

$$\text{cardinality}\{\mu < \sigma : \dim(\mu)+1 = \dim(\sigma) ;\ f(\mu) \geq f(\sigma)\} \leq 1$$

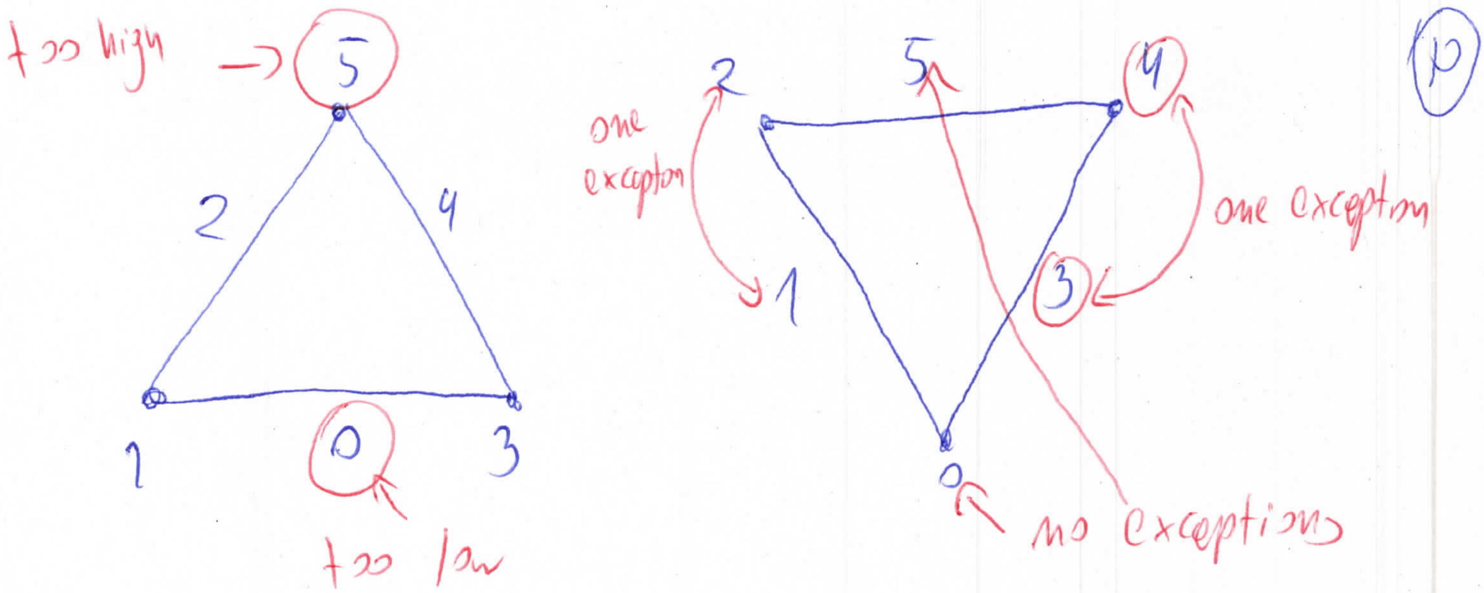
number of co-dimension-1 cells μ in the boundary of σ that have the same, or higher value of f then σ is at most 1.

Intuitively speaking, f is a Discrete Morse Function if for every cell σ, f at boundary element of σ is <u>below</u> $f(\sigma)$ with <u>at most one exception</u> and if for every cell τ having σ in the boundary $f(\tau)$ is <u>above</u> $f(\sigma)$ with at most one exception.

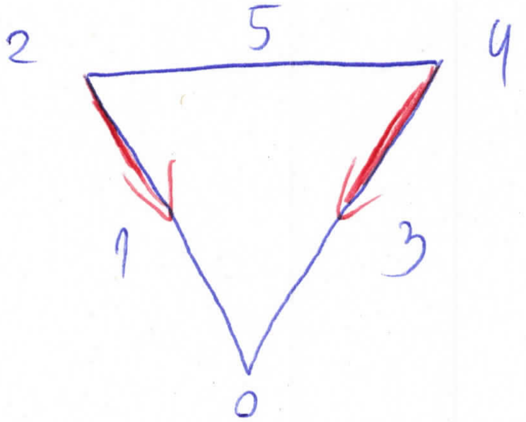
<u>Eg</u>: Which of these is a Discrete Morse Function?



Answer in the next page...



For cells for which the one exception happens we can define a pairing from that cell to the one below, i.e.

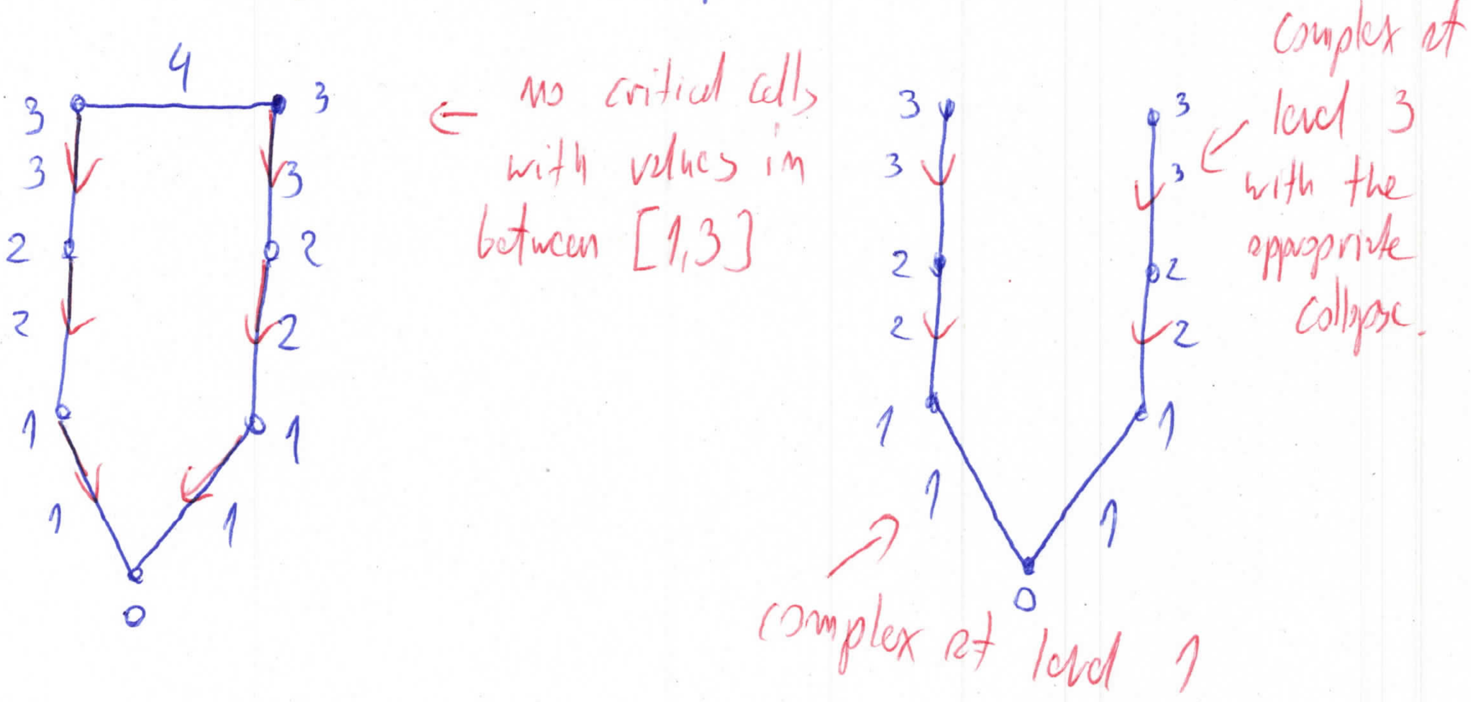


Those pairings corresponds to the gradient vector field, and are almost perfect analogy $\to$ those are the cells for which there is a unique direction of a steepest descent.

What about the cells that do not have exception? (11)
They are called critical cells & once again, they correspond to critical ~~cells~~ points in continuous Morse Theory.

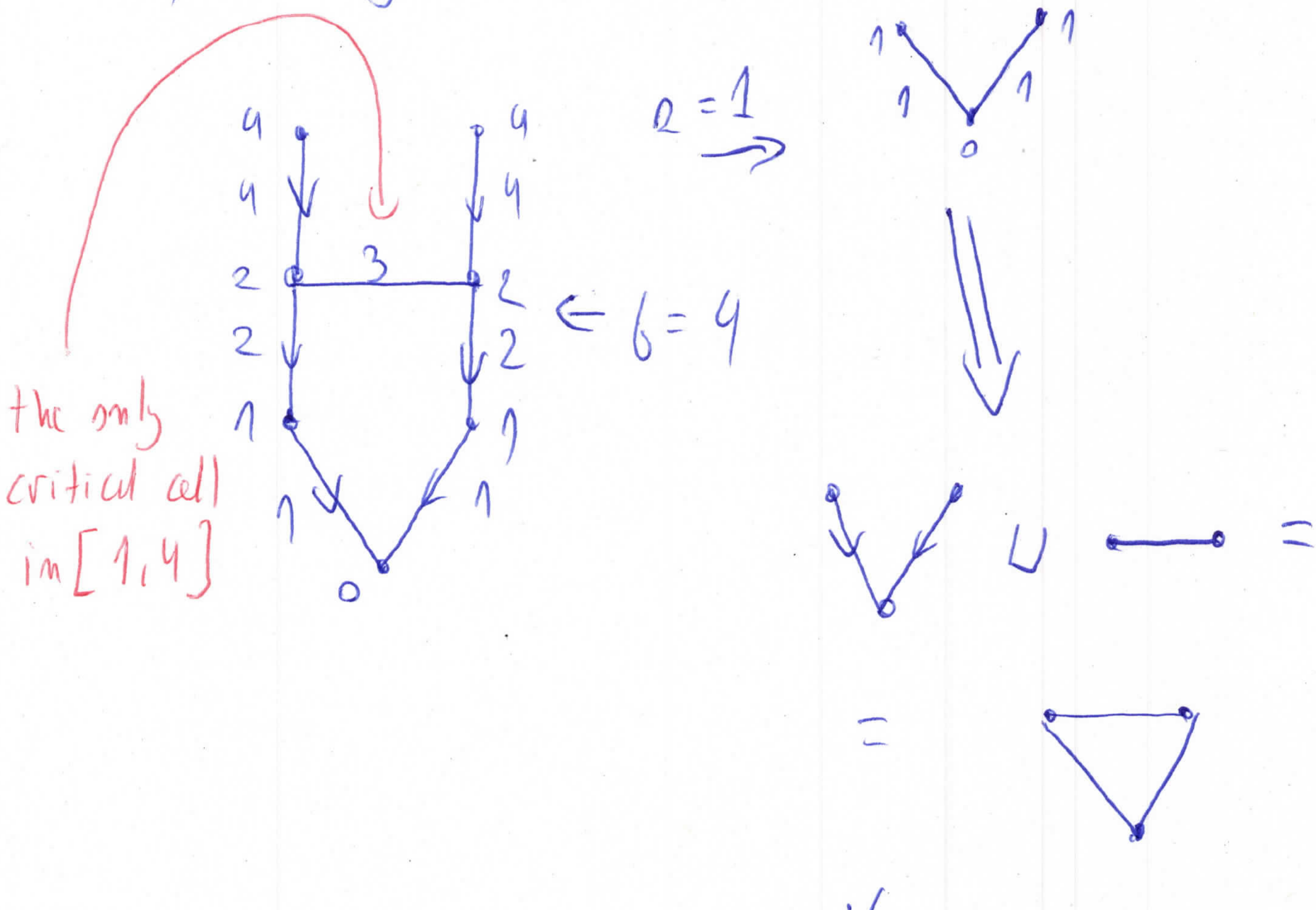
The analogy is almost 1-1. For example the following Theorem holds:

Theorem
If there are no critical cells with $f(\sigma) \in [a,b]$, then ~~cells~~ complex composed of cells at the level ~~a~~ a & at level b are homotopy equivalent.
Moreover, the gradient path of the Discrete Morse function gives this retraction explicitly:



We also have an analogous of gluing the cells;

If σ is unique critical cell in $f^{-1}([a,b])$, then the complex at the level b can be obtained from the complex at the level a after gluing (according to the discrete gradient) a cell of the same dimension as σ; Eg;



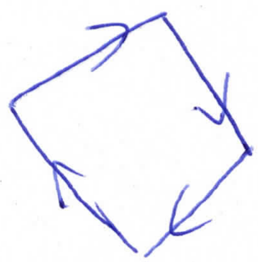
<u>Moreover</u>: Given a cell complex K with a Discrete Morse Function f, it is homotopically equivalent to a ~~chain~~ CW complex with exactly one cell ~~of~~ for critical cell of K with f.

Moreover, this CW complex can be constructed explicitly - which is easy when it comes to idea, but a bit hard formally.

Here is an idea: Before we get there let us have one simple observation:

→ We do not use the exact values of the Discrete Morse function, just ones gradient. Therefore from now on, we will only display gradient & "forget" about the values.

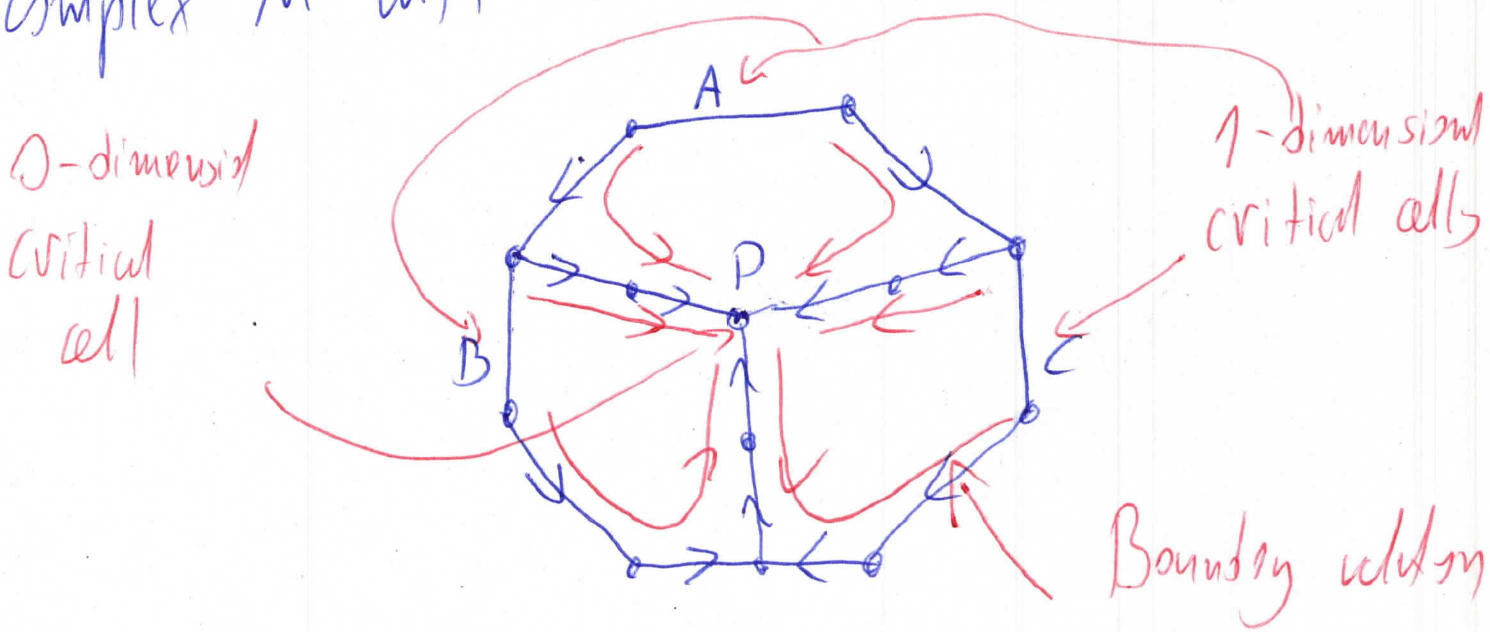
→ Because gradient vector field is coming from a function, it have to be curl free ⇒ i.e. no closed loops:



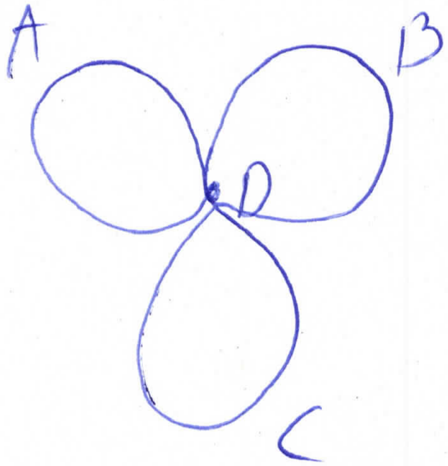
this is not a vector field.

→ For a gradient vector field one cell can be paired to AT MOST one other cell

Let us finally get the complex that is equivalent to ~~a Morse complex~~ the initial complex M with a discrete Morse function on it.



$\simeq$



Discrete Morse theory is used to speed up persistence computations in Perseus software by Vidit Nanda.

Discrete Morse theory vs Persistence

homotopy homology

give connections by gradient, but no pairings. give pairings

Hard cases in both theories are in somewhat correspondence [Joint work with TU Berlin].

Some fun facts about relation of continuous & discrete Morse theory:

(1) For S^n, $n \geq 3$ there exist smooth Morse function with two critical cells.

But

There exis a triangulation of S^n with at least k critical cells for every Discrete Morse function, for every k.

But...

2) Benedetti 2010;

M - n dimensional manifold with a continuous morse function having c_i points of the index i. Let us have arbitrary triangulation of M. Then there exist $k \geq 0$ such that k-th baricentric subdivision of the triangulation of M admits a discrete Morse function with c_i critical cells in dimension i

i.e. when we have freedom to chose triangulation, or at least to subdivide it, then discrete theory is almost equivalent to the continuous one.

3) J.H.C. Whitehead, 1939 -

For a manifold M, if there ~~is~~ exist a triangulation of M that collapses to a point, then M is homeomorphic ~~with~~ to a ball

But...

(9) Adiprasito, Benedetti 2011,
For every $n \geq 5$, there exist n-dimensional manifold M that is not homeomorphic to a ball, but admit a triangulation collapsible to a point.

(I have not explained the relation between collapsibility & discrete Morse theory, but I assume that it can be deduced).

Further reading:
Robin Forman, User's guide to Discrete Morse Theory
Nick Scoville, Discrete Morse Theory
Dmitry Kozlov, Combinatorial Algebraic Topology